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Spectral analysis of self-affine functions

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By definition, a self-affine function is a continuous function $f:[0,1] \rightarrow \mathbb{R}$ with $f(0)=0$, $f(x) \neq 0$ such that there exist a real number α with $0 < \alpha \leq 1$, an integer $r \geq 2$ and finitely many functions $g_1, g_2, \dots, g_k: [0,1] \rightarrow \mathbb{R}$ satisfying that

(1) for any $N=0,1,2,\dots$ and $j=0,1,\dots,r^N-1$, there exists $h=1,2,\dots,k$ such that

$$(\#) \quad r^{\alpha N} (f((j+x)r^{-N}) - f(jr^{-N})) = g_h(x) \quad (\forall x \in [0,1]), \text{ and}$$

(2) for any $n=0,1,2,\dots$, $\ell=0,1,\dots,r^n-1$ and $h=1,2,\dots,k$, there exist integers N and j with $N \geq n$ and $\ell r^{N-n} \leq j < (\ell+1)r^{N-n}$ such that $(\#)$ holds.

The above α and r are called an order and a base of f , respectively. The notion of self-affine function was introduced by N. Kôno [1] in a restrictive sense, which was generalized by the author [2] as above. It is known that

I. if f is a self-affine function of order α , then f satisfies the Hölder continuity of order exactly α at each point [1],

hence, α is determined by f ,

II. a self-affine function is a linear function if and only if it has an order 1,

III. if a self-affine function f has bases r and ℓ with $\log r / \log \ell$ irrational, then f is a linear function, and

IV. there exist exactly a countably infinite number of self-affine functions which are linearly independent.

A non-linear, self-affine function f has a remarkable spectral property, namely, the Fourier coefficient $\hat{f}(n)$ decreases in the order of n^{-1} as $n \rightarrow \infty$, where

$$\hat{f}(n) := \int_0^1 (f(x) - xf(1)) e^{-2\pi i n x} dx \quad (n \in \mathbb{Z}).$$

For a self-affine function f with order α and base r , define

$$\tau_{r,N}(j) = r^{\alpha N} (f(j+1)r^{-N} - f(jr^{-N})),$$

where $N=0,1,2,\dots$ and $j=0,1,\dots,r^N-1$. Note that

$$\{\tau_{r,N}(j); N=0,1,2,\dots, j=0,1,\dots,r^N-1\}$$

is a finite set. A self-affine function f with base r is called r -invertible if

$$\tau_{r,N}(j) = \tau_{r,N}(j')$$

holds for any $N=0,1,2,\dots$ and $j=0,1,\dots,r^N-1$, where j' is an integer with $0 \leq j' < r^N$ such that

$$[j'r^{-n}] - r[j'r^{-n-1}] = [jr^{N-n-1}] - r[jr^{N-n}]$$

for any $n=0,1,\dots,N-1$.

THEOREM. Let f be a self-affine function with order α and base r . Then,

$$\hat{f}(n) = \frac{1}{2\pi i n} \lim_{N \rightarrow \infty} r^{-\alpha N} \sum_{j=0}^{r^N-1} \tau_{r,N}(j) e^{-2\alpha i j n r^{-N}}$$

holds for any $n \in \mathbb{Z}$ with $n \neq 0$. Moreover, if f is r -invertible, then we have

$$\hat{f}(rn) = r^{-1} \hat{f}(n)$$

for any $n \in \mathbb{Z}$ with $n \neq 0$.

Example 1. Let

$$f(x) = \sum_{n=1}^{\infty} 3^{-n} \sum_{j=1}^n x_{2j-1} 2^{j-1} (x_{2n}),$$

where

$$x = \sum_{n=1}^{\infty} x_n 3^{-n}$$

with $x_n \in \{0,1,2\}$ and η is a permutation $\begin{pmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \end{pmatrix}$. Then, f is a 9-invertible self-affine function of order $1/2$, which shall be called Peano function. We have

$$\hat{f}(n) = \frac{1}{i\pi n} \prod_{k=1}^{\infty} \frac{1}{3} \frac{1 + \omega_k^n + \omega_k^{2n}}{1 - \omega_k^n + \omega_k^{2n}},$$

where $\omega_k = e^{2\pi i 9^{-k}}$ and $n \neq 0$.

Example 2. Let

$$f(x) = \sum_{n=1}^{\infty} 2^{-[n/2]} (-1)^{\sum_{j=1}^{n-2} x_j x_{j+1}} x_n,$$

where

$$x = \sum_{n=1}^{\infty} x_n 2^{-n}$$

with $x_n \in \{0,1\}$. Then, f is a 4-invertible self-affine function with order $1/2$, which shall be called Rudin-Shapiro function.

We have

$$\hat{f}(n) = \frac{1}{2\pi i n} \lim_{N \rightarrow \infty} 2^{-N} \sum_{j=0}^{4^N-1} a_j e^{-2\pi i j n 4^{-N}}$$

for any $n \neq 0$, where

$$a_j = (-1)^{\sum_{k=0}^{\infty} j_k j_{k+1}}$$

with

$$j = \sum_{k=0}^{\infty} j_k 2^k \quad (j_k \in \{0,1\}).$$

REFERENCES

- [1] Norio Kôno, On self-affine functions, Japan Journal of Applied Mathematics 3-2 (1986), pp. 259-269
- [2] Teturo Kamae, A characterization of self-affine functions, ibid., pp. 271-280